J. Fluid Mech. (1969), *vol.* 38, *part* 3, *pp.* 547–564
Printed in Great Britain

Hypersonic weak-interaction solutions for flow past a very slender axisymmetric body

By ARTHUR K. CROSS AND WILLIAM B. BUSH
University of Southern California, Department of Aerospace Engineering

(Received 5 March 1969)

The Navier–Stokes hypersonic weak-interaction theory is presented for the flow of a viscous, heat-conducting, compressible fluid past a very slender axisymmetric body, when the ratio of the radius of the body to the radial thickness of the viscous region, produced and supported by the body, is much less than unity. The fluid is assumed to be a perfect gas having constant specific heats, a constant Prandtl number of order unity, and viscosity coefficients varying as a power of the absolute temperature. Solutions are studied for the free-stream Mach number, the free-stream Reynolds number based on the axial length of the body, and the reciprocal of the weak-interaction parameter much greater than unity.

It is shown that, for the viscosity–temperature exponent ω less than 1, seven distinct layers span the region between the shock wave and the body, which is of arbitrary shape. The leading approximations for the behaviour of the flow in these seven layers are analyzed, and the restrictions imposed on the theory are obtained.

1. Introduction

This paper presents the formulation of the hypersonic weak-interaction theory (HWIT) régime for flow past a very slender axisymmetric body, for the case when the transverse body curvature is much less than the radial thickness of the viscous region supported by the body. Thus, for this presentation, it is assumed that the free-stream Mach number M, and the free-stream Reynolds number based on the axial distance of the body R_L, are much greater than unity, and that the body radial thickness parameter δ_b, and the viscous region radial thickness parameter δ, are much less than unity, with the parameters ordered by

$$\delta_b \ll \delta \ll 1/M \ll 1.$$

From the above, it is seen that the present formulation is an extension of the existing formulations of the HWIT axisymmetric problem (cf. for example, Probstein 1955; Probstein & Elliot 1956), which are ordered by

$$\delta \leqslant \delta_b \ll 1/M \ll 1.$$

In the following sections, the uniformly valid solutions of the flow variables in the domain between the shock wave and body are determined by a rigorous and complete treatment of the seven distinct physical layers that span the domain. The formulation provides the orders of magnitude for all the flow quantities in each region, and thus affords physical insight into the flow field picture.

The Navier–Stokes equations in cylindrical polar co-ordinates are given in § 2.

In § 3 the analysis of the principal inviscid layer (PIL), supported by the viscous layers acting as a slender 'effective body' with a thickness ratio of $O(\delta)$, is given. This analysis, which is just the linearized supersonic inviscid flow theory in a modified hypersonic form, closely follows that given by Bush & Cross (1967) for the HWIT for flow past a flat plate. The PIL solutions presented are those valid when the 'effective body' is a paraboloid. These solutions, however, are non-uniform near the shock wave and the outer edge of the 'effective body'.

In § 4 an analysis of the exterior inviscid layer (EIL), the thin non-linear layer intermediate to the shock wave and the PIL, provides the solutions for the flow variables which are uniform at the shock and match to the PIL solutions. In the process, the shock shape correction produced by a paraboloidal 'effective body' is determined. The analysis for the EIL, again, parallels that presented by Bush & Cross (1967) for the flat plate HWIT.

In § 5 an analysis is presented for the interior inviscid layer (IIL), which is introduced to remove the non-uniformities in the PIL solutions in the vicinity of the 'effective body'. The treatment for the IIL, whose radial thickness is of the same order of magnitude as that of the 'effective body', is a hypersonic modification of the incompressible treatment due to Cole (1968).

In § 6 the 'Oseen-like' principal viscous layer (PVL), whose radial thickness is $O(\delta)$, is analyzed. The formulation for this layer follows those of Stewartson (1964) and Bush (1968) for the corresponding layer in the hypersonic strong-interaction theory (HSIT) régime. The asymptotic behaviours of the PVL solutions near the layer's 'sharp' paraboloidal outer edge and near the body are determined; they are given in terms of the logarithmic radial variable first introduced by Bush (1968). As in the HSIT case, these HWIT PVL solutions are not capable of satisfying the non-slip, temperature-specified boundary conditions at the body surface. Consequently, the PVL must be complemented by an additional layer, interior to the PVL, in order to describe the adjustment of the flow in the vicinity of the body surface.

This layer interior to the PVL, a 'Couette-like' body viscous layer (BVL) with dissipation, is investigated in § 7, with the analysis again paralleling that of Bush (1968) for the equivalent layer of the HSIT régime. The BVL solutions match the PVL solutions and satisfy the axial velocity non-slip and the temperature-specified boundary conditions at the body surface, but do not satisfy the radial velocity non-slip condition.

The failure of the BVL radial velocity solution to satisfy the non-slip boundary condition at the body leads to the introduction of a 'Stokes-like' layer directly adjacent to the body, designated as the wall viscous layer (WVL). An analysis of this WVL, in § 8, leads to solutions which match to the BVL solutions and completely satisfy the surface boundary conditions. The skin-friction and heat-transfer coefficients, for an arbitrary body shape, are evaluated (to two terms) from the WVL solutions.

Since all the IIL and PVL solutions do not match directly, in the appendix an analysis of the required viscous transition layer (VTL), intermediate to these two layers, is presented. The asymptotic behaviours, for the flow variables in this VTL, as the PVL and IIL are approached, are determined, using essentially the same techniques as Bush & Cross (1967). The VTL solutions are, then, shown to match these of the PVL and IIL.

A schematic diagram of the hypersonic weak-interaction layers, for flow past a very slender axisymmetric body, is given in figure 1.

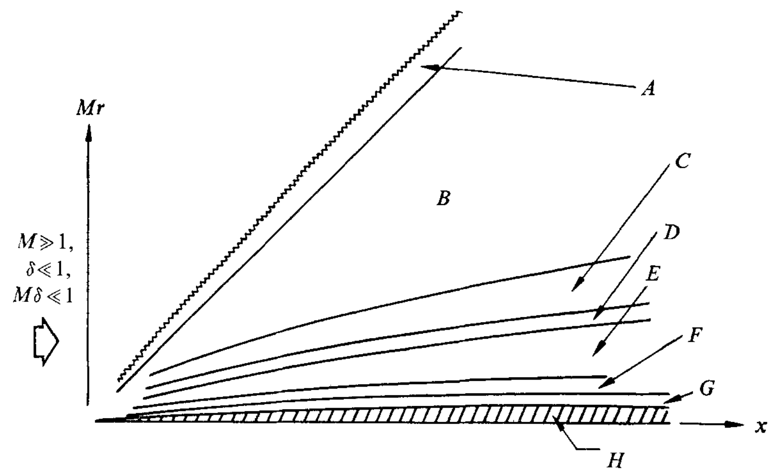

FIGURE 1. Schematic diagram of hypersonic weak interaction layers for flow past a very slender axisymmetric body. ∿∿∿∿, shock wave; ——, Mach wave. A, exterior inviscid layer $r-(1/M)\,x = O((M\delta)^{\frac{4}{3}}/M)$. B, principal inviscid layer $r = O(1/M)$. C, interior inviscid layer $r = O(\delta)$. D, viscous transition layer $\Delta r = O(1/\sqrt{R_L})$. E, principal viscous layer $r = O(\delta)$. F, body viscous layer $r = O(\delta_b^{t_b}\delta^{1-t_b})$, $O < t_b < 1$. G, wall viscous layer $r = O(\delta_b)$. H, very slender body $r = \delta_b R_b(x)$.

2. The equations of motion

Consider the flow of a viscous, compressible gas past a very slender axisymmetric body. Let $x_1 = Lx$ and $r_1 = Lr$ represent the cylindrical polar coordinates along the axis of symmetry from the vertex of the body and normal to this axis, respectively. The length L is chosen so that x is of order unity in the region where the weak-interaction theory is valid.

Under this formulation, the equation of the surface of this slender body is

$$r = \delta_b R_b(x), \quad \text{with} \quad \delta_b = \delta D_b \ll \delta \ll 1, \quad R_b(x) = O(1),$$

where δ represents the scaling of the effective thickness of the viscous layer(s) supported by the body. The velocity components in the x_1- and r_1-directions are $u_1 = u_\infty u$ and $v_1 = u_\infty v$, and the pressure, temperature, and density, respectively, are $p_1 = p_\infty p$, $T_1 = T_\infty T$, and $\rho_1 = \rho_\infty \rho$, where u_∞, p_∞, T_∞ and ρ_∞, respectively, are the velocity in the x_1-direction, pressure, temperature, and density in the undisturbed region upstream of the body.

A perfect gas ($p = \rho T$) is assumed, having (i) constant specific heats, c_{v_1} and c_{p_1}, with $\gamma = (c_{p_1}/c_{v_1}) = \text{const.}$; (ii) a constant Prandtl number of order unity ($\sigma = \text{const.} = O(1)$); and (iii) its first and second viscosity coefficients proportional to a power, ω, of the absolute temperature ($\mu_1 = \mu_\infty \mu = \mu_\infty T^\omega$, with $\frac{1}{2} \leqslant \omega < 1$; $\lambda_1 = \mu_\infty \lambda = j\mu_\infty \mu = j\mu_\infty T^\omega$, $j = \text{const.} = O(1)$).

The Navier–Stokes equations of motion in cylindrical polar co-ordinates for the flow of such a gas are

$$\frac{\partial}{\partial x}(\rho u r)+\frac{\partial}{\partial r}(\rho v r) = 0, \tag{2.1}$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}\right)+\frac{\epsilon}{\theta_S}\frac{\partial p}{\partial x}$$
$$= \frac{1}{R_L}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rT^{\omega}\left\{\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right\}\right)+\frac{\partial}{\partial x}\left(T^{\omega}\left\{(2+j)\frac{\partial u}{\partial x}+\frac{j}{r}\frac{\partial}{\partial r}(rv)\right\}\right)\right], \tag{2.2}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}\right)+\frac{\epsilon}{\theta_S}\frac{\partial p}{\partial r}$$
$$= \frac{1}{R_L}\left[\frac{\partial}{\partial r}\left(T^{\omega}\left\{(2+j)\frac{\partial v}{\partial r}+j\frac{v}{r}+j\frac{\partial u}{\partial x}\right\}\right)+2T^{\omega}\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{\partial}{\partial x}\left(T^{\omega}\left\{\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right\}\right)\right], \tag{2.3}$$

$$\rho\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r}\right)-\frac{2\epsilon}{1+\epsilon}\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial r}\right)$$
$$= \frac{1}{\sigma}\frac{1}{R_L}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rT^{\omega}\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial x}\left(T^{\omega}\frac{\partial T}{\partial x}\right)\right]$$
$$+\frac{2}{1+\epsilon}\frac{\theta_S}{R_L}\left[T^{\omega}\left\{\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right)^2+2\left[\left(\frac{\partial v}{\partial r}\right)^2+\left(\frac{v}{r}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]\right\}\right]$$
$$+\frac{2j}{1+\epsilon}\frac{\theta_S}{R_L}\left[T^{\omega}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{\partial u}{\partial x}\right\}^2\right], \tag{2.4}$$

where $\epsilon = (\gamma-1)/(\gamma+1) = O(1)$, the Newtonian approximation (of $\epsilon \ll 1$) not being invoked; $M^2 = (\rho_\infty u_\infty^2/\gamma p_\infty) \gg 1$; $\theta_S = \epsilon\{(1+\epsilon)/(1-\epsilon)\}M^2 \gg 1$; and $R_L = (\rho_\infty u_\infty L/\mu_\infty) \gg 1$.

3. The principal inviscid (shock) layer

According to hypersonic weak-interaction theory (HWIT), the slender body, whose surface is given by $r = \delta_b R_b(x)$, combines with the thin viscous, heat-conducting layer(s) at the body surface to disturb the uniform external flow. This combination of the body and the viscous layer(s), whose outer edge is given by $r = \delta R_k(x)$, with δ, the thickness parameter of this combination (for the flow régime under consideration, $\delta_b = \delta D_b \ll \delta \ll 1/M \ll 1$) acts as slender 'effective body', producing an oblique (Rankine–Hugoniot) shock wave and an inviscid 'shock layer' between the shock wave and the 'effective body'. The analysis of the HWIT principal inviscid (shock) layer provides the starting point for this investigation.

For this principal inviscid layer (PIL), the distorted co-ordinates and flow quantities have the following representations (cf. Bush & Cross 1967):

$$x_a = x, \quad r_a = Mr; \tag{3.1}$$

$$u = 1+(K_a/M^2)\,u_a+\dots, \quad v = (K_a/M)\,v_a+\dots,$$
$$p = 1+K_a p_a+\dots, \quad \rho = 1+K_a\rho_a+\dots, \quad T = 1+K_a T_a+\dots, \tag{3.2}$$

where K_a = parameter to be determined in § 5 $\ll 1$, and $f_a = f_a(x_a, r_a) = O(1)$.

Substitution of (3.1) and (3.2) into the equations of motion, (2.1)–(2.4), yields, to leading approximation, the following linear system:

$$p_a-(\rho_a+T_a) = 0, \quad r_a\frac{\partial}{\partial x_a}(\rho_a)+\frac{\partial}{\partial r_a}(r_a v_a) = 0,$$

$$\frac{\partial u_a}{\partial x_a}+\frac{1}{\gamma}\frac{\partial p_a}{\partial x_a} = 0, \quad \frac{\partial v_a}{\partial x_a}+\frac{1}{\gamma}\frac{\partial p_a}{\partial r_a} = 0, \quad \frac{\partial T_a}{\partial x_a}-\frac{\gamma-1}{\gamma}\frac{\partial p_a}{\partial x_a} = 0. \tag{3.3}$$

The ratio of the orders of magnitude of the leading viscosity and heat-conduction terms (which have been neglected) to those of the inviscid convection and pressure gradient terms (which have been retained) is

$$(M^2/R_L) = (M^{2(1+\omega)}\theta_k^{1+\omega}/R_L\delta^2)(\delta^2/\theta_k^{1+\omega})(1/M^{2\omega}).$$

It is demonstrated, in § 7, that $(M^2/R_L)\to 0$.

With rearrangement, (3.3) becomes

$$\frac{\partial^2 v_a}{\partial x_a^2}-\frac{\partial}{\partial r_a}\left(\frac{\partial v_a}{\partial r_a}+\frac{v_a}{r_a}\right) = 0, \quad \frac{\partial^2 p_a}{\partial x_a^2}-\frac{1}{r_a}\frac{\partial}{\partial r_a}\left(r_a\frac{\partial p_a}{\partial r_a}\right) = 0,$$

$$u_a = -\frac{1}{\gamma}p_a+S_a(r_a), \quad T_a = \frac{\gamma-1}{\gamma}p_a+H_a(r_a), \quad \rho_a = \frac{1}{\gamma}p_a-H_a(r_a). \tag{3.4}$$

Subject to subsequent verification by matching, the following solutions to (3.4) are proposed:

$$u_a = -U_{a,0}[(x_a^2-r_a^2)^{\frac{1}{2}}]^{-1}, \tag{3.5a}$$

$$v_a = U_{a,0}x_a[r_a(x_a^2-r_a^2)^{\frac{1}{2}}]^{-1}, \tag{3.5b}$$

$$p_a = \gamma S_a-\gamma u_a, \quad T_a = (\gamma-1)S_a+H_a-(\gamma-1)u_a, \quad \rho_a = S_a-H_a-u_a, \tag{3.5c, d, e}$$

where $U_{a,0}$ = const. (to be determined in § 5). These solutions are derived from the hypersonic limit of the Kármán–Moore potential for flow past a paraboloid,

$$\phi_a = U_{a,0}\log[\{(x_a^2-r_a^2)^{\frac{1}{2}}-x_a\}/r_a]. \tag{3.6}$$

In the vicinity of the shock wave, where $(x_a-r_a) = r_a^*\to 0$, x_a fixed, the asymptotic solutions for the flow quantities, from (3.5), are found to exhibit singular behaviours, i.e.

$$v_a, \quad -u_a, \quad p_a/\gamma, \quad T_a/(\gamma-1), \quad \rho_a = U_{a,0}(2x_a r_a^*)^{-\frac{1}{2}}+\ldots\to\infty. \tag{3.7†}$$

Hence, in order to develop uniformly valid solutions for the layers spanning the region between the shock wave and the body surface, a thin exterior inviscid layer (EIL) is introduced between the PIL and the Rankine–Hugoniot shock (cf. Bush & Cross 1967). The analysis of the EIL is presented in § 4. Through this analysis the functions $S_a(r_a)$ and $H_a(r_a)$ are determined.

In the vicinity of the outer edge of the 'effective body', where $r_a\to 0$, x_a fixed,

$$u_a = -U_{a,0}/x_a+\ldots, \quad v_a = U_{a,0}/r_a+\ldots,$$

$$p_a = \gamma S_a(0)+\gamma U_{a,0}/x_a+\ldots, \quad T_a = (\gamma-1)S_a(0)+H_a(0)+(\gamma-1)U_{a,0}/x_a+\ldots,$$

$$\rho_a = S_a(0)-H_a(0)+U_{a,0}/x_a+\ldots. \tag{3.8}$$

† It follows, then, that the asymptotic behaviour of $r_a^*(x_a, v_a)$, as $v_a\to\infty$, x_a fixed, is: $r_a^* = (\frac{1}{2}U_{a,0}^2)(x_a v_a^2)^{-1}+\ldots\to 0$.

From (3.8), it is seen that the behaviour of v_a is not uniform as the 'effective body' is approached. The analysis of the interior inviscid layer (IIL), in which the above non-uniformity in the radial velocity component is removed, is presented in § 5.

4. The inviscid exterior layer

The weak-interaction shock relations are now considered. The shock wave in the weak-interaction limit ($M\delta = K \to 0$) corresponds to a small disturbance (of a magnitude to be determined) on a Mach wave. Thus, in this limit, the shock shape is taken to be

$$r_{sh}(x) = (1/M)\,[x + K_f F(x) + \ldots], \tag{4.1}$$

where K_f = parameter to be determined (cf. §§ 4 and 5) $\ll 1$. For such a shock shape, the flow variables at the downstream face of the shock are

$$(1-u_{sh})/(K_f/M^2) = v_{sh}/(K_f/M) = (p_{sh}-1)/\gamma K_f = (T_{sh}-1)/(\gamma-1)K_f = (\rho_{sh}-1)/K_f$$
$$= \{4/(\gamma+1)\}\,F'(x) + \ldots. \tag{4.2}$$

The forms of (3.5) and of (4.1)–(4.2) suggest that the proper representations for the co-ordinates and flow quantities in this layer adjacent to the shock front are

$$x_f = x, \quad r_f = (x - Mr)/K_f; \tag{4.3}$$

$$u = 1 + (K_f/M^2)\,u_f + \ldots, \quad v = (K_f/M)\,v_f + \ldots,$$
$$p = 1 + K_f p_f + \ldots, \quad T = 1 + K_f T_f + \ldots, \quad \rho = 1 = K_f \rho_f + \ldots, \tag{4.4}$$

where $f_f = f_f(x_f, r_f) = O(1)$. The shock relations for these representations are

$$(v_f)_{sh} = -(u_f)_{sh} = (\rho_f)_{sh} = (p_f)_{sh}/\gamma = (T_f)_{sh}/(\gamma-1) = \{4/(\gamma+1)\}\,F'(x), \tag{4.5}$$

where $(f_f)_{sh} = f_f(x_f, -F(x_f))$.

Substitution of (4.3) and (4.4) into the equations of motion, (2.1)–(2.4), yields the following quasilinear system:

$$\frac{\partial v_f}{\partial x_f} + \frac{1}{\gamma}\left(\frac{\partial p_f}{\partial x_f} - \frac{\gamma+1}{\gamma} p_f \frac{\partial p_f}{\partial r_f}\right) + \frac{v_f}{r_f} = 0,$$
$$p_f - (\rho_f + T_f) = 0, \quad p_f - \gamma\rho_f = 0,$$
$$\frac{\partial}{\partial r_f}\left(u_f + \frac{1}{\gamma} p_f\right) = 0, \quad \frac{\partial}{\partial r_f}\left(v_f - \frac{1}{\gamma} p_f\right) = 0. \tag{4.6}$$

Application of the shock relations, (4.5), to (4.6), shows that

$$p_f/\gamma = \rho_f = -u_f = T_f/(\gamma-1) = v_f, \tag{4.7}$$

where the governing equation for v_f is the quasilinear equation,

$$\frac{\partial v_f}{\partial x_f} - \frac{\gamma+1}{2} v_f \frac{\partial v_f}{\partial r_f} + \frac{v_f}{2x_f} = 0. \tag{4.8†}$$

† The last term of (4.8) represents the three-dimensional axisymmetric contribution; only the first two terms are present in the corresponding HWIT flat plate equation.

The general solution of (4.8) is

$$r_f = -(\gamma+1)\,v_f x_f + \Phi(v_f^2 x_f). \tag{4.9}$$

However, the assumption of a self-similar solution, with the variables

$$v_f = x_f^{m-1} V_f(\zeta_f), \quad \zeta_f = r_f/x_f^m, \tag{4.10}$$

reduces (4.8) to

$$[(\gamma+1)\,V_f + 2m\zeta_f]\frac{dV_f}{d\zeta_f} + (1-2m)\,V_f = 0. \tag{4.11}$$

The solution of this equation, in terms of the original variables, is

$$r_f = -(\gamma+1)\,v_f x_f + N_f(v_f^2 x_f)^{-m/(1-2m)}, \tag{4.12}$$

where N_f = const. determined subsequently. It is noteworthy that the second term in (4.12) represents the restricted similarity form of the function $\Phi(v_f^2 x_f)$ in the general solution of (4.8).

The procedure for matching the PIL and EIL solutions follows that employed in the HWIT flat plate analysis (Bush & Cross 1967). A comparison of (3.7), and of (4.7) and (4.12), indicates that the solutions of these two layers match if

$$m = \tfrac{1}{3}, \quad K_f = (K_a)^{\frac{2}{3}}, \quad N_f = U_{a,0}^2/2, \quad S_a(r_a) = H_a(r_a) = 0. \tag{4.13}$$

Evaluating (4.12) at the shock, by means of (4.5) and (4.13), and solving the resulting equation for the shock shape correction, $F(x)$, yields

$$F(x) = (\tfrac{3}{2})\,[(\gamma+1)\,U_{a,0}/2]^{\frac{2}{3}} x^{\frac{1}{3}}. \tag{4.14}$$

From results obtained in §5, $K_a = (M\delta)^2$ and $U_{a,0} = \frac{1}{2}A_k^2$. Accordingly, the shock shape can be expressed as

$$(r)_{sh} = (1/M)\,\{x + (M\delta)^{\frac{4}{3}}\,[\tfrac{3}{2}\{(\gamma+1)\,A_k^2/4\}^{\frac{2}{3}}]\,x^{\frac{1}{3}} + \ldots\}. \tag{4.15}$$

5. The interior inviscid layer

Inspection of (3.1) and (3.5) shows that the behaviour of v_a is not uniform as the effective body is approached, since, as $r_a \to 0$, $v_a \to (U_{a,0}/r_a) \to \infty$. Accordingly, a layer intermediate to the principal inviscid layer and the 'effective body' produced by the viscous effects is introduced (cf. Cole 1968). It is anticipated that this layer will have the same thickness ordering δ as the 'effective body', to permit evaluation of the 'effective body' boundary conditions in terms of a finite value of the layer's independent variable $r_i = r/\delta$.

These considerations, the forms of the solutions of (3.5) and of the 'effective body' boundary conditions, suggest that the proper representations for the co-ordinates and flow quantities for this layer adjacent to the 'effective body' are

$$x_i = x, \quad r_i = r/\delta; \tag{5.1}$$

$$u = 1 + (K_a/M^2)\,u_i + \ldots, \quad v = \delta v_i + \ldots,$$

$$p = 1 + K_a p_i + \ldots, \quad T = 1 + K_a T_i + \ldots, \quad \rho = 1 + K_a \rho_i + \ldots, \tag{5.2}$$

where $f_i = f_i(x_i, r_i) = O(1)$.

With these representations, for $(1/R_L\delta^2) \ll 1$ (as is shown in § 7), the leading terms in the equations of motion are

$$p_i = \rho_i + T_i, \quad \frac{\partial}{\partial r_i}(r_i v_i) = 0,$$

$$\frac{\partial u_i}{\partial x_i} + v_i\frac{\partial u_i}{\partial r_i} + \frac{1}{\gamma}\frac{\partial p_i}{\partial x_i} = 0, \quad \frac{\partial v_i}{\partial x_i} + v_i\frac{\partial v_i}{\partial r_i} + \frac{1}{\gamma}\frac{\partial p_i}{\partial r_i} = 0,$$

$$\frac{\partial T_i}{\partial x_i} + v_i\frac{\partial T_i}{\partial r_i} - \frac{\gamma-1}{\gamma}\left(\frac{\partial p_i}{\partial x_i} + v_i\frac{\partial p_i}{\partial r_i}\right) = 0. \tag{5.3}$$

Taking the new independent and dependent variables to be

$$\xi_i = x_i, \quad \eta_i = r_i/x_i^{\frac{1}{2}}, \tag{5.4}$$

$$u_i = U_i(\eta_i)/\xi_i, \quad v_i = V_i(\eta_i)/\xi_i^{\frac{1}{2}},$$

$$p_i = P_i(\eta_i)/\xi_i, \quad T_i = \Theta_i(\eta_i)/\xi_i, \quad \rho_i = D_i(\eta_i)/\xi_i, \tag{5.5}$$

leads to the following system of ordinary differential equations describing the self-similar flow contemplated:

$$P_i = D_i + \Theta_i, \quad \frac{d}{d\eta_i}(\eta_i V_i) = 0,$$

$$\left(\{2V_i - \eta_i\}\frac{d}{d\eta_i} - 2\right)U_i - \frac{1}{\gamma}\left(\eta_i\frac{d}{d\eta_i} + 2\right)P_i = 0,$$

$$\left(\{2V_i - \eta_i\}\frac{d}{d\eta_i} - 1\right)V_i + \frac{2}{\gamma}\frac{dP_i}{d\eta_i} = 0,$$

$$\left(\{2V_i - \eta_i\}\frac{d}{d\eta_i} - 2\right)\left(\Theta_i - \frac{\gamma-1}{\gamma}P_i\right) = 0. \tag{5.6}$$

This system of equations is easily integrated. In terms of the original variables, the IIL solutions, which match those proposed for the PIL, are taken to be

$$v = \delta\left[\frac{U_{a,0}}{\xi_i^{\frac{1}{2}}\eta_i}\right] + \ldots, \quad u = 1 - (K_a/M^2)\left[\frac{U_{a,0}}{\xi_i}\right] + \ldots,$$

$$p = 1 + K_a\left[\frac{\gamma U_{a,0}}{\xi_i}\left(1 - \frac{U_{a,0}}{2\eta_i^2}\right)\right] + \ldots, \quad T = 1 + K_a\left[\frac{(\gamma-1)U_{a,0}}{\xi_i}\left(1 - \frac{U_{a,0}}{2\eta_i^2}\right)\right] + \ldots, \tag{5.7}$$

with

$$K_a = (M\delta)^2. \tag{5.8}$$

The 'effective body' boundary condition that the velocity normal to the outer edge of this 'effective body', $r = \delta R_k$, be zero, is to be satisfied in terms of the IIL flow variables. The 'effective body' is taken to be a paraboloid, as will be verified in § 6, so that

$$R_k = A_k x^{\frac{1}{2}}. \tag{5.9}$$

(It is noted that the proposed PIL solutions anticipated the taking of the 'effective body' to be a paraboloid.) Thus, for the paraboloidal 'effective body', it is required that

$$v_i \to (A_k/2)\,x_i^{-\frac{1}{2}} \quad \text{as} \quad r_i \to A_k x_i^{\frac{1}{2}} \Rightarrow V_i \to (A_k/2) \quad \text{as} \quad \eta_i \to A_k. \tag{5.10}$$

To satisfy this condition, it follows that

$$U_{a,0} = \tfrac{1}{2}A_k^2. \tag{5.11}$$

The value that A_k, itself, must take is determined in §7.

For future reference, the asymptotic behaviours of the IIL solutions, as $\eta_i \to A_k$, are given:

$$u = 1-\delta^2\left[\frac{A_k^2}{2\xi_i}\right]+\dots, \quad v = \delta\left[\frac{A_k}{2\xi_i^{\frac{1}{2}}}\left\{1-\left(\frac{\eta_i-A_k}{A_k}\right)+\dots\right\}\right]+\dots,$$

$$p = 1+(M\delta)^2\left[\frac{3\gamma A_k^2}{8\xi_i}\left\{1+\frac{2}{3}\left(\frac{\eta_i-A_k}{A_k}\right)+\dots\right\}\right]+\dots,$$

$$T = 1+(M\delta)^2\left[\frac{3(\gamma-1)A_k^2}{8\xi_i}\left\{1+\frac{2}{3}\left(\frac{\eta_i-A_k}{A_k}\right)+\dots\right\}\right]+\dots. \tag{5.12}$$

6. The principal viscous layer

The principal viscous layer (PVL) is a high temperature, low density region, across which the pressure is constant. This layer has a 'sharp outer edge', $r = \delta R_k(x)$, and acts as a slender 'effective body' producing the weak oblique (Rankine–Hugoniot) shock wave and the inviscid shock layer between the shock and the 'body'. The analysis of this major viscous region (which is much larger than the actual body: $M\delta_b \ll M\delta \ll 1$) provides the starting point for the analysis of the viscous zone.

For the analysis of this layer, the following distorted co-ordinates and flow variable expansions are assumed:

$$x_k = x, \quad r_k = r/\delta; \tag{6.1}$$

$$u = 1+\theta_k u_k+\dots, \quad v = \delta v_k+\dots,$$

$$T = M^2\theta_k T_k+\dots, \quad p = 1+(M\delta)^2 P_k+\dots, \tag{6.2}$$

with θ_k = parameter to be determined (in §7) $\ll 1$, and $f_k = f_k(x_k, r_k) = O(1)$.

With these representations, for

$$\Gamma = (M^2\theta_k)^{1+\omega}/R_L\delta^2 = O(1), \quad 1/M^2 \ll \theta_k \ll 1, \tag{6.3}$$

the leading terms in the equations of motion are:

$$\frac{\partial}{\partial x_k}\left(\frac{r_k}{T_k}\right)+\frac{\partial}{\partial r_k}\left(\frac{r_k v_k}{T_k}\right) = 0,$$

$$\frac{\partial u_k}{\partial x_k}+v_k\frac{\partial u_k}{\partial r_k} = \Gamma\frac{T_k}{r_k}\frac{\partial}{\partial r_k}\left(r_k T_k^{\omega}\frac{\partial u_k}{\partial r_k}\right),$$

$$\frac{\partial p_k}{\partial r_k} = 0 \Rightarrow p_k = P_k(x_k),$$

$$\frac{\partial T_k}{\partial x_k}+v_k\frac{\partial T_k}{\partial r_k} = \frac{\Gamma}{\sigma}\frac{T_k}{r_k}\frac{\partial}{\partial r_k}\left(r_k T_k^{\omega}\frac{\partial T_k}{\partial r_k}\right). \tag{6.4}$$

Consider the case where the outer edge of the principal viscous layer is taken to be a paraboloidal surface of the form $r_k = R_k(x_k) = A_k x_k^{\frac{1}{2}}$. Then, subject to the outer edge boundary conditions,

$$u_k \to 0, \quad T_k \to 0, \quad v_k \to (A_k/2)\,x_k^{-\frac{1}{2}}, \quad p_k \to (3\gamma A_k^2/8)\,x_k^{-1}, \tag{6.5}$$

as $r_k \to A_k x_k^{\frac{1}{2}}$, (6.4) may be reduced to a system of ordinary differential equations.

For a paraboloidal PVL outer edge, and for a formulation parallel to that presented by Bush (1968) for the corresponding layer in the HSIT, the appropriate independent and dependent variables are

$$s_k = x_k, \quad t_k = \log\{A_k x_k^{\frac{1}{2}}/r_k\} \tag{6.6a}$$

(such that $t_k \to 0$ as $r_k \to R_k = A_k x_k^{\frac{1}{2}}$, $t_k \to \log(1/D_b) + \log(R_k/R_b) \to \infty$ as $r_k \to D_b R_b$);

$$u_k = U_k(t_k), \quad v_k = (A_k/2)\, s_k^{-\frac{1}{2}} e^{-t_k} V_k(t_k),$$
$$T_k = \Theta_k(t_k), \quad p_k = P_k(s_k) = (3\gamma A_k^2/8)\, s_k^{-1}. \tag{6.6b}$$

In terms of the variables of (6.6), (6.4) and (6.5) become

$$\frac{d}{dt_k}\left(\frac{1-V_k}{\Theta_k}\right) - 2\left(\frac{1-V_k}{\Theta_k}\right) = -\frac{2}{\Theta_k}, \tag{6.7a}$$

$$\frac{d}{dt_k}\left(\Theta_k^{\omega}\frac{dU_k}{dt_k}\right) = (A_k^2/2\Gamma)\, e^{-2t_k}\left(\frac{1-V_k}{\Theta_k}\right)\frac{dU_k}{dt_k}, \tag{6.7b}$$

$$\frac{d}{dt_k}\left(\Theta_k^{\omega}\frac{d\Theta_k}{dt_k}\right) = (\sigma A_k^2/2\Gamma)\, e^{-2t_k}\left(\frac{1-V_k}{\Theta_k}\right)\frac{d\Theta_k}{dt_k}; \tag{6.7c}$$

$$U_k, \quad \Theta_k \to 0, \quad V_k \to 1, \quad \text{as} \quad t_k \to 0. \tag{6.8}$$

Further, it is found that, in conjunction with (6.7a) and (6.8), (6.7b) and (6.7c) yield, upon integration,

$$\Theta_k^{\omega}\frac{dU_k}{dt_k} = (A_k^2/2\Gamma)\left[2\int_0^{t_k}(U_k/\Theta_k)\, e^{-2\nu}\, d\nu + (U_k/\Theta_k)(1-V_k)\, e^{-2t_k}\right], \tag{6.9a}$$

$$\Theta_k^{\omega}\frac{d\Theta_k}{dt_k} = (\sigma A_k^2/2\Gamma)\,[1 - V_k\, e^{-2t_k}]. \tag{6.9b}$$

Near the outer edge of the PVL, where $t_k \to 0$, it is found that, for

$$(1-\omega) = O(1) > 0,$$

the asymptotic solutions for the flow quantities in (6.7) are

$$\Theta_k = \Theta_{k,0}\, t_k^{2/(1+\omega)} + \ldots \Rightarrow T = M^2\theta_k[\Theta_{k,0}\, t_k^{2/(1+\omega)} + \ldots] + \ldots,$$
$$U_k = U_{k,0}\, t_k^{q} + \ldots \quad \Rightarrow u = 1 + \theta_k[U_{k,0}\, t_k^{q} + \ldots] + \ldots,$$
$$V_k = 1 - V_{k,0}\, t_k + \ldots \Rightarrow v = \delta(A_k/2)\, s_k^{-\frac{1}{2}}[1 - (V_{k,0}+1)\, t_k + \ldots] + \ldots, \tag{6.10a}$$

where

$$\Theta_{k,0} = \left[2\left(\frac{1+\omega}{1-\omega}\right)\left(\frac{\sigma A_k^2}{2\Gamma}\right)\right]^{1/(1+\omega)},$$

$$U_{k,0} = \text{undetermined const.}, \quad q = \frac{1}{\sigma} + \frac{1-\omega}{1+\omega},$$

$$V_{k,0} = \left(2\,\frac{1+\omega}{1-\omega}\right). \tag{6.10b}$$

A comparison of (5.12) and (6.10) shows clearly that the functional behaviours of the temperature solutions for the IIL and the PVL, as $\eta_i \to A_k$ and $t_k \to 0$, respectively, do not permit direct matching between these two layers, and the introduction of an intermediate transition layer (see appendix) is necessary to complete the matching.

Near the inner edge of the PVL, where $t_k \to \infty$, it is seen, from (6.9), that

$$\Theta_k^\omega \frac{dU_k}{dt_k} \to \frac{A_k^2}{2\Gamma}\int_0^\infty (U_k/\Theta_k)\, e^{-2t_k}\, dt_k; \quad \Theta_k^\omega \frac{d\Theta_k}{dt_k} \to \frac{\sigma A_k^2}{2\Gamma}.$$

Hence, the shear and heat-conduction terms are found to dominate the momentum and energy equations, respectively, yielding, in this limit, the following asymptotic solutions of the similarity equations:

$$\begin{aligned}
\Theta_k &= \Theta_{k,\infty} t_k^{1/(1+\omega)} + \ldots \Rightarrow T = M^2\theta_k[\Theta_{k,\infty} t_k^{1/(1+\omega)} + \ldots] + \ldots,\\
U_k &= U_{k,\infty} t_k^{1/(1+\omega)} + \ldots \Rightarrow u = 1 + \theta_k[U_{k,\infty} t_k^{1/(1+\omega)} + \ldots] + \ldots,\\
V_k &= V_{k,\infty} t_k^{-1} + \ldots \quad \Rightarrow v = \delta(A_k/2)\, s_k^{-\frac{1}{2}} e^{-t_k}\, [V_{k,\infty} t_k^{-1} + \ldots] + \ldots,
\end{aligned} \tag{6.11a}$$

where

$$\Theta_{k,\infty} = \left[(1+\omega)\left(\frac{\sigma A_k^2}{2\Gamma}\right)\right]^{1/(1+\omega)},$$

$$U_{k,\infty} = \frac{1}{\sigma}\Theta_{k,\infty}\int_0^\infty (U_k/\Theta_k)\, e^{-2t_k}\, dt_k,$$

$$V_{k,\infty} = [2(1+\omega)]^{-1}. \tag{6.11b}$$

Consider the introduction of new independent variables, s_k^* and t_k^*, defined by

$$s_k^* = s_k, \quad t_k^* = t_k - G_k(s_k), \quad \text{with} \quad G_k(s_k) = \log\{R_k(s_k)/R_b(s_k)\}. \tag{6.12}$$

Then, in terms of these new variables, for s_k^* fixed, $t_k^* \to \infty$, (6.11) may be expressed as

$$\begin{aligned}
T &= M^2\theta_k\left[\Theta_{k,\infty}\left\{t_k^{*1/(1+\omega)} + \frac{G_k}{1+\omega} t_k^{*-\omega/(1+\omega)} + \ldots\right\} + \ldots\right] + \ldots,\\
u &= 1 + \theta_k\left[U_{k,\infty}\left\{t_k^{*1/(1+\omega)} + \frac{G_k}{1+\omega} t_k^{*-\omega/(1+\omega)} + \ldots\right\} + \ldots\right] + \ldots,\\
v &= \delta\left[\frac{R_b}{2s_k^*} V_{k,\infty} t_k^{*-1} e^{-t_k^*} + \ldots\right] + \ldots.
\end{aligned} \tag{6.13}$$

The PVL solutions of (6.11) and/or (6.13) do not satisfy the usual non-slip, temperature-specified boundary conditions at the body surface, namely,

$$u, v \to 0 \quad T/T_{\text{stag}} \to \varphi_w = \text{specified fnc}(s_k), \quad \text{as} \quad t_k \to \log(1/D_b) + G_k(s_k) \to \infty. \tag{6.14}$$

Consequently, in accordance with the approach of Bush (1968), the analysis of an additional layer, interior to the PVL, in which $t_k = O\{\log(1/D_b)\} \to \infty$, is introduced in the next section, in order to describe the adjustment of the flow in the vicinity of the body surface.

7. The body viscous layer

The body viscous layer (BVL) is introduced to investigate the flow in the region near the body. The analysis of this HWIT region follows the approach employed by Bush (1968) for the corresponding HSIT region. It is anticipated from the previous section that this region should be dominated by viscous and heat-conduction effects. From the solutions of the PVL flow quantities for $t_k \to \infty$, it is expected that, near the outer edge of the BVL, the flow variables should have the following behaviours: $u \to 1$, $T/M^2 \to 0$, $v/\delta_b \to \infty$, $p \to 1$. The analysis of the BVL is carried out in the distorted co-ordinate system adopted by Bush (1968), namely:

$$s_b = x, \quad t_b = \tau_b \log\{\delta R_b(x)/r\} = \tau_b t_k^*, \tag{7.1}$$

where
$$\tau_b = [\log(1/D_b)]^{-1} = [\log(\delta/\delta_b)]^{-1} \to 0. \tag{7.2}$$

Hence,

$$t_b \to 1 \quad \text{as} \quad r \to \delta_b R_b; \quad t_b \to \tau_b \log(R_b/R_k) = -\tau_b G_k \to 0 \quad \text{as} \quad r \to \delta R_k.$$

The expansions for the flow quantities in this layer are

$$u = u_b + \tau_b u_b^{(1)} + \ldots, \quad v = \delta\tau_b e^{-t_b/\tau_b} v_b + \ldots,$$
$$T = M^2(T_b + \tau_b T_b^{(1)} + \ldots), \quad p = 1 + (M\delta)^2 p_b + \ldots, \tag{7.3}$$

with $f_b = f_b(s_b, t_b) = O(1)$.

For $\Gamma = O(1)$ and $(\tau_b/M^2\theta_k^{1+\omega}) \ll 1$, the leading terms in the equations of motion for the BVL are:

$$\frac{\partial}{\partial s_b}\left(\frac{u_b}{T_b}\right) = 0 \Rightarrow \frac{u_b}{T_b} = C_b(t_b), \tag{7.4a}$$

$$v_b = -\frac{T_b}{2}\left[R_b'\frac{\partial}{\partial t_b}\left(\frac{u_b}{T_b}\right) + R_b\left(\frac{u_b}{T_b}\right)\frac{\partial}{\partial s_b}\left(\frac{u_b^{(1)}}{u_b} - \frac{T_b^{(1)}}{T_b}\right)\right], \ \ldots; \tag{7.4b}$$

$$\frac{\partial}{\partial t_b}\left(T_b^\omega \frac{\partial u_b}{\partial t_b}\right) = 0, \tag{7.5a}$$

$$\frac{\partial}{\partial t_b}\left(T_b^\omega\left[\omega\frac{T_b^{(1)}}{T_b}\frac{\partial u_b}{\partial t_b} + \frac{\partial u_b^{(1)}}{\partial t_b}\right]\right) = 0, \ \ldots; \tag{7.5b}$$

$$\frac{\partial p_b}{\partial t_b} = 0 \Rightarrow p_b = P_b(s_b) = (3\gamma A_k^2/8)\, s_b^{-1}, \ \ldots; \tag{7.6}$$

$$\frac{\partial}{\partial t_b}\left(T_b^\omega\frac{\partial T_b}{\partial t_b}\right) + \sigma(\gamma-1)\,T_b^\omega\left(\frac{\partial u_b}{\partial t_b}\right)^2 = 0, \tag{7.7a}$$

$$\frac{\partial}{\partial t_b}\left(T_b^\omega\left[\omega\frac{T_b^{(1)}}{T_b}\frac{\partial T_b}{\partial t_b} + \frac{\partial T_b^{(1)}}{\partial t_b}\right]\right) + \sigma(\gamma-1)\,T_b^\omega\frac{\partial u_b}{\partial t_b}\left[\omega\frac{T_b^{(1)}}{T_b}\frac{\partial u_b}{\partial t_b} + 2\frac{\partial u_b^{(1)}}{\partial t_b}\right] = 0, \ \ldots. \tag{7.7b}$$

Consider first (7.4a), (7.5a), and (7.7a). The first integral of (7.5a) is taken to be

$$T_b^\omega\frac{\partial u_b}{\partial t_b} = -S_b, \tag{7.8a}$$

where $S_b = S_b(s_b) > 0$. The first integral of (7.7a) is taken to be

$$T_b^\omega\frac{\partial T_b}{\partial t_b} = \pm[2\sigma(\gamma-1)]^{\frac{1}{2}}\,S_b(T_{b,m} - T_b)^{\frac{1}{2}}, \tag{7.8b}$$

where $T_b \leqslant T_{b,m} = T_{b,m}(s_b) > 0$. The positive branch of (7.8b) represents the temperature field between the point of maximum temperature ($T_b \to T_{b,m}$, $t_b \to t_{b,m}$) and the outer edge of the BVL ($T_b \to 0$, $t_b \to 0$); the negative branch of

(7.8*b*) represents the temperature field between the point of maximum temperature ($T_b \to T_{b,m}$, $t_b \to t_{b,m}$) and the body surface ($T_b \to T_{b,w}$, $t_b \to 1$).

For $u_b \to 1$, $T_b \to 0$ as $t_b \to 0$, and $u_b \to 0$, $T_b \to T_{b,w}$ as $t_b \to 1$, (7.8*a*) and (7.8*b*) combine to yield the following 'Crocco-like' relation between T_b and u_b:

$$T_b = T_{b,w}(1-u_b)(1+\Omega u_b); \quad \Omega = \sigma(\gamma-1)/2T_{b,w}. \tag{7.9a}$$

In determining (7.9*a*), it is found that

$$T_{b,m}/T_{b,w} = (\Omega+1)^2/4\Omega. \tag{7.9b}$$

However, in order to satisfy (7.4*a*), it follows, from (7.8) and (7.9), that it is necessary to require that

$$u_b = u_b(t_b), \quad T_b = T_b(t_b); \quad T_{b,w} \text{ (and } T_{b,m}), \quad S_b = \text{consts.} \tag{7.10}$$

Note that when (7.8*b*) is integrated to determine $T_b(t_b)$, the solution for $u_b(t_b)$ follows directly from the integration of (7.8*a*).

The shear function S_b for $u_b \to 1$ as $t_b \to 0$, $u_b \to 0$ as $t_b \to 1$, by direct integration of (7.8*a*), with $T_b(u_b; T_{b,w})$ given by (7.9*a*), may be expressed as a quadrature:

$$S_b/T^{\omega}_{b,w} = S^*_b = \int_0^1 [(1-\nu)(1+\Omega\nu)]^{\omega}\, d\nu. \tag{7.11}$$

Consider, now, (7.4*b*), (7.5*b*), and (7.7*b*). From (7.5*b*) and (7.7*b*), taking into account the solutions for u_b and T_b, it is found that the solutions for $u^{(1)}_b$ and $T^{(1)}_b$ may be expressed as

$$u^{(1)}_b = G_b \frac{du_b}{dt_b} - H_b(1-u_b) + J_b, \tag{7.12a}$$

$$T^{(1)}_b = G_b \frac{dT_b}{dt_b} + 2H_b T_b, \tag{7.12b}$$

with G_b, H_b, $J_b = \text{fncs}(s_b)$ to be determined. Indeed, for $u^{(1)}_b$, $T^{(1)}_b \to 0$ as $t_b \to 1$, it is required that

$$H_b = -\left[\frac{1}{2T_{b,w}}\left(\frac{dT_b}{dt_b}\right)_{,w}\right] G_b = [\tfrac{1}{2}(\Omega-1)\, S^*_b]\, G_b = H_{b,0} G_b, \tag{7.13a}$$

$$J_b = -\left[\left(\frac{du_b}{dt_b}\right)_{,w}\right] G_b + H_b = [\tfrac{1}{2}(\Omega+1)\, S^*_b]\, G_b = J_{b,0} G_b. \tag{7.13b}$$

In turn, it follows that, from (7.12) and (7.13), (7.4*b*) becomes

$$v_b = -\frac{1}{2}\left[(R'_b + R_b G'_b)\left(\frac{du_b}{dt_b} - \frac{u_b}{T_b}\frac{dT_b}{dt_b}\right) - H_{b,0} R_b G'_b u_b + (J_{b,0} - H_{b,0}) R_b G'_b\right]. \tag{7.14}$$

Near the outer edge of the BVL, where $t_b \to 0$, it is found that the asymptotic solutions for the flow quantities in (7.4)–(7.7) are

$$T_b = \Theta_{b,0} t_b^{1/(1+\omega)} + \dots, \quad T^{(1)}_b = \frac{G_b}{1+\omega}\, \Theta_{b,0} t_b^{-\omega/(1+\omega)} + \dots \Rightarrow$$

$$T = M^2\left\{[\Theta_{b,0} t_b^{1/(1+\omega)} + \dots] + \tau_b\left[\frac{G_b}{1+\omega}\, \Theta_{b,0} t_b^{-\omega/(1+\omega)} + \dots\right] + \dots\right\},$$

$$u_b = 1 + U_{b,0} t_b^{1/(1+\omega)} + \ldots, \quad u_b^{(1)} = \frac{G_b}{1+\omega} U_{b,0} t_b^{-\omega/(1+\omega)} + \ldots \Rightarrow$$

$$u = [1 + U_{b,0} t_b^{1/(1+\omega)} + \ldots] + \tau_b \left[\frac{G_b}{1+\omega} U_{b,0} t_b^{-\omega/(1+\omega)} + \ldots \right] + \ldots,$$

$$v_b = V_{b,0}(R_b' + R_b G_b') t_b^{-1} + \ldots \Rightarrow$$

$$v = \delta \tau_b e^{-t_b/\tau_b} [V_{b,0}(R_b' + R_b G_b') t_b^{-1} + \ldots] + \ldots, \tag{7.15a}$$

where

$$\Theta_{b,0} = [(1+\omega)(\Omega+1) S_b^*]^{1/(1+\omega)} T_{b,w},$$

$$U_{b,0} = -(1+\omega) S_b^* (\Theta_{b,0}/T_{b,w})^{-\omega},$$

$$V_{b,0} = [2(1+\omega)]^{-1}. \tag{7.15b}$$

A comparison of the asymptotic solutions of (6.13) and (7.15) indicates that the PVL and BVL solutions match if

$$\theta_k = \tau_b^{1/(1+\omega)}; \tag{7.16a}$$

$$\Theta_{k,\infty} = \Theta_{b,0}, \quad U_{k,\infty} = U_{b\,0}; \tag{7.16b}$$

$$G_b = G_k = \log(R_k/R_b). \tag{7.16c}$$

It is seen that the above conditions for matching do not require the specification of the (actual) body shape. From (7.16a), if Γ is taken to be identically equal to one, then it follows that the PVL thickness parameter δ is defined by

$$\delta = M^{1+\omega} \tau_b^{\frac{1}{2}} / R_L^{\frac{1}{2}} \ll 1/M \ll \tau_b^{\frac{1}{2}(1+\omega)} \ll 1, \tag{7.17a}†$$

where $\tau_b(M, R_L, \delta_b)$ is given (approximately) by

$$\tau_b = [\log(M^{1+\omega}/R_L^{\frac{1}{2}} \delta_b)]^{-1} \left\{ 1 + \frac{1}{2} \frac{\log[\log(M^{1+\omega}/R_L^{\frac{1}{2}} \delta_b)]}{[\log(M^{1+\omega}/R_L^{\frac{1}{2}} \delta_b)]} + \ldots \right\} \ll 1. \tag{7.17b}$$

Moreover, from the above, the body thickness parameter δ_b must satisfy

$$\delta_b \ll (M^{1+\omega}/R_L^{\frac{1}{2}}) \exp(-M^{2(2+\omega)}/R_L). \tag{7.18}$$

From (6.11b), (7.15b) and (7.16b), with $\Gamma = 1$, the PVL shape constant A_k is determined to be

$$A_k = [(2/\sigma)(\Omega+1) T_{b,w}^{1+\omega} S_b^*]^{\frac{1}{2}} = \mathrm{fnc}(T_{b,w}; \sigma, \gamma, \omega). \tag{7.19}$$

Now consider the asymptotic solutions for the flow quantities in the BVL as $t_b \to 1$. In terms of the original variables, these solutions are:

$$T = M^2\{[T_{b,w} + \Theta_{b,1}(1-t_b) + \Theta_{b,1}^*(1-t_b)^2/2 + \ldots] + \tau_b[\Theta_{b,1}^{(1)}(1-t_b) + \ldots] + \ldots\},$$

$$u = [U_{b,1}(1-t_b) + U_{b,1}^*(1-t_b)^2/2 + \ldots] + \tau_b[U_{b,1}^{(1)}(1-t_b) + \ldots] + \ldots,$$

$$v = \delta_b \tau_b \exp\{(1-t_b)/\tau_b\}\{[V_{b,1} + V_{b,1}^*(1-t_b) + \ldots] + \ldots\}, \tag{7.20a}$$

† The requirements of the PIL and IIL that (M^2/R_L), $(1/R_L\,\delta^2) \to 0$ are satisfied, since $M^2/R_L \ll 1/R_L\,\delta^2 = 1/M^{2(1+\omega)}\tau_b \ll 1$.

where

$$\Theta_{b,1} = -\left(\frac{dT_b}{dt_b}\right)_{,w}, \quad \Theta^*_{b,1} = \left(\frac{d^2T_b}{dt_b^2}\right)_{,w}, \quad \Theta^{(1)}_{b,1} = [T^{-1}_{b,w}\Theta^2_{b,1} - \Theta^*_{b,1}]\,G_b,$$

$$U_{b,1} = -\left(\frac{du_b}{dt_b}\right)_{,w}, \quad U^*_{b,1} = \left(\frac{d^2u_b}{dt_b^2}\right)_{,w}, \quad U^{(1)}_{b,1} = [(2T_{b,w})^{-1}U_{b,1}\Theta_{b,1} - U^*_{b,1}]\,G_b,$$

$$V_{b,1} = (U_{b,1}/2)\,R'_b, \quad V^*_{b,1} = -[(4T_{b,w})^{-1}U_{b,1}\Theta_{b,1}]\{2(1+\omega)\,R'_b + (1+2\omega)\,R_b G'_b\}. \tag{7.20b}$$†

Rewriting (7.20) in terms of the independent variables

$$\xi_b = s_b, \quad \eta_b = \exp\{(1-t_b)/\tau_b\} = r/\delta_b R_b$$

yields

$$T = M^2\{[T_{b,w} + \Theta_{b,1}(\tau_b \log\eta_b) + \Theta^*_{b,1}(\tau_b\log\eta_b)^2/2 + \ldots] + \tau_b[\Theta^{(1)}_{b,1}(\tau_b\log\eta_b) + \ldots] + \ldots\},$$
$$u = [U_{b,1}(\tau_b\log\eta_b) + U^*_{b,1}(\tau_b\log\eta_b)^2/2 + \ldots] + \tau_b[U^{(1)}_{b,1}(\tau_b\log\eta_b) + \ldots] + \ldots,$$
$$v = \delta_b\tau_b\eta_b\{[V_{b,1} + V^*_{b,1}(\tau_b\log\eta_b) + \ldots] + \ldots\}. \tag{7.21}$$

From either (7.20) or (7.21), it is seen that, although u and $\{(T/M^2) - T_{b,w}\} \to 0$ as $t_b \to 1$ (i.e. $r \to \delta_b R_b$), $v \to$ a small finite value as $t_b \to 1$. Consequently, the BVL solutions do not satisfy all the non-slip, temperature-specified boundary conditions at the body surface (cf. (6.14)). Therefore, in § 8, a thin layer, interior to the BVL and adjacent to the body surface, is introduced, a layer in which the flow adjusts itself, so that all the surface boundary conditions are satisfied.

8. The wall viscous layer

The wall viscous layer (WVL) is introduced to study the flow in a region next to the body. For this layer, the proper representations for the co-ordinates and the flow quantities are

$$\xi_\beta = x, \quad \eta_\beta = r/\delta_b R_b; \tag{8.1}$$

$$u = \tau_b(u_\beta + \tau_b u^{(1)}_\beta + \ldots), \quad v = \delta_b\tau_b(v_\beta + \tau_b v^{(1)}_\beta + \ldots),$$
$$T = M^2[T_{b,w} + \tau_b(T_\beta + \tau_b T^{(1)}_\beta + \ldots)], \quad p = 1 + (M\delta)^2 p_\beta + \ldots. \tag{8.2}$$

With these representations, the leading terms in the equations of motion are

$$\eta_\beta\left(R_b\frac{\partial}{\partial\xi_\beta} - R'_b\eta_\beta\frac{\partial}{\partial\eta_\beta}\right)u_\beta + \frac{\partial}{\partial\eta_\beta}(\eta_\beta v_\beta) = 0, \tag{8.3a}$$

$$\eta_\beta\left(R_b\frac{\partial}{\partial\xi_\beta} - R'_b\eta_\beta\frac{\partial}{\partial\eta_\beta}\right)\left\{u^{(1)}_\beta - \frac{T_\beta}{T_{b,w}}u_\beta\right\} + \frac{\partial}{\partial\eta_\beta}\left(\eta_\beta\left\{v^{(1)}_\beta - \frac{T_\beta}{T_{b,w}}v_\beta\right\}\right) = 0, \ \ldots; \tag{8.3b}$$

† In (7.20b),

$$\left(\frac{du_b}{dt_b}\right)_w = -S^*_b = -U_{b,1}, \quad \left(\frac{dT_b}{dt_b}\right)_{,w} = -(\Omega-1)\,T_{b,w}S^*_b = -\Theta_{b,},$$

$$\left(\frac{d^2u_b}{dt_b^2}\right)_{,w} = -\omega(\Omega-1)\,S^{*2}_b = U^*_{b,1} = -\omega T^{-1}_{b,w}U_{b,1}\Theta_{b,1},$$

$$\left(\frac{d^2T_b}{dt_b^2}\right)_{,w} = -\{\omega(\Omega-1)^2 + 2\Omega\}\,T_{b,w}S^{*2}_b = \Theta^*_{b,1} = -T^{-1}_{b,w}\{\omega\Theta^2_{b,1} + 2\Omega T^2_{b,w}U^2_{b,1}\}.$$

$$\frac{\partial}{\partial \eta_\beta}\left(\eta_\beta \frac{\partial u_\beta}{\partial \eta_\beta}\right) = 0, \tag{8.4a}$$

$$\frac{\partial}{\partial \eta_\beta}\left(\eta_\beta \left\{\frac{\partial u_\beta^{(1)}}{\partial \eta_\beta} + \omega \frac{T_\beta}{T_{b,w}} \frac{\partial u_\beta}{\partial \eta_\beta}\right\}\right) = 0, \ \ldots; \tag{8.4b}$$

$$\frac{\partial p_\beta}{\partial \eta_\beta} = 0, \ \ldots; \tag{8.5}$$

$$\frac{\partial}{\partial \eta_\beta}\left(\eta_\beta \frac{\partial T_\beta}{\partial \eta_\beta}\right) = 0, \tag{8.6a}$$

$$\frac{\partial}{\partial \eta_\beta}\left(\eta_\beta \left\{\frac{\partial T_\beta^{(1)}}{\partial \eta_\beta} + \omega \frac{T_\beta}{T_{b,w}} \frac{\partial T_\beta}{\partial \eta_\beta}\right\}\right) + \sigma(\gamma-1)\,\eta_\beta \left(\frac{\partial u_\beta}{\partial \eta_\beta}\right)^2 = 0, \ \ldots. \tag{8.6b}$$

To satisfy the non-slip, temperature-specified surface boundary conditions, it is necessary that

$$u_\beta, \quad u_\beta^{(1)}, \quad v_\beta, \quad v_\beta^{(1)}, \quad T_\beta, \quad T_\beta^{(1)} \to 0 \quad \text{as} \quad \eta_\beta \to 1. \tag{8.7}$$

The solutions of (8.3)–(8.6), in terms of the original flow variables, satisfying the boundary conditions of (8.7) and, in addition, matching to the BVL solutions (cf. (7.21)), are determined to be

$$u = \tau_b[U_{b,1}(\log \eta_\beta)] + \tau_b^2[U_{b,1}^*(\log \eta_\beta)^2/2 + U_{b,1}^{(1)}(\log \eta_\beta)] + \ldots,$$

$$v = \delta_b\left\{\tau_b\left[V_{b,1}\left(\frac{\eta_\beta^2-1}{\eta_\beta}\right)\right] + \tau_b^2\left[V_{b,1}^*(\eta_\beta \log \eta_\beta) + V_{\beta,1}\left(\frac{\eta_\beta^2-1}{\eta_\beta}\right) + V_{\beta,1}^*(\log \eta_\beta/\eta_\beta)\right] + \ldots\right\},$$

$$T = M^2\{T_{b,w} + \tau_b[\Theta_{b,1}(\log \eta_\beta)] + \tau_b^2[\Theta_{b,1}^*(\log \eta_\beta)^2/2 + \Theta_{b,1}^{(1)}(\log \eta_\beta)] + \ldots\},$$

$$p = 1 + (M\delta)^2[(3\gamma A_k^2/8)\,\xi_\beta^{-1}] + \ldots, \tag{8.8a}$$

where

$$V_{\beta,1} = [(8T_{b,w})^{-1}U_{b,1}\Theta_{b,1}]\{2(1+2\omega)\,R_b' G_b + (1+2\omega)\,R_b G_b' + 2(2+\omega)\,R_b'\},$$

$$V_{\beta,1}^* = -[(2T_{b,w})^{-1}U_{b,1}\Theta_{b,1}]\,R_b'. \tag{8.8b}$$

The skin friction coefficient C_f, and the heat transfer coefficient C_q, from the WVL solutions, are

$$C_f = 2[\mu_1(\partial u_1/\partial r_1)]_{,w}/\rho_\infty u_\infty^2$$
$$= 2\left(\frac{M^{2\omega}\tau_b}{R_L\delta_b}\right)\frac{S_b}{R_b}\left[1 + \tau_b\left(\frac{1+2\omega}{2}\right)(\Omega-1)\,S_b^* G_b + \ldots\right]; \tag{8.9a}$$

$$C_q = 2[k_1(\partial T_1/\partial r_1)]_{,w}/\rho_\infty u_\infty^3$$
$$= \left(\frac{M^{2\omega}\tau_b}{R_L\delta_b}\right)\frac{S_b}{R_b}\left(\frac{\Omega-1}{\Omega}\right)[1 + \tau_b\{(1+\omega)(\Omega-1)^2 + 2\Omega\}(\Omega-1)^{-1}S_b^* G_b + \ldots]. \tag{8.9b}$$

The work reported in this paper was supported, in part, by NASA, Grant NGR-05-018-044.

Appendix. The viscous transition layer

A viscous transition layer (VTL), intermediate to the IIL and PVL, is introduced to permit uniformly valid solutions for the flow variables. For this VTL, the distorted co-ordinates and flow variables are taken to have the following representations:

$$\xi_t = x, \quad \eta_t = [(r/\delta R_k)-1]/\delta_t = [(r/\delta A_k x^{\frac{1}{2}})-1]/\delta_t, \quad \delta_t \ll 1; \tag{A 1}$$

$$\begin{aligned} u &= 1+(\theta_k \delta_t^q u_t+\ldots)+\delta^2 w_t+\ldots, \quad \delta^2 \ll \theta_k\delta_t^q \ll \delta_t \ll 1,\\ v &= \delta R_k'(1+\delta_t v_t+\ldots) = \delta(A_k/2)\,\xi_t^{-\frac{1}{2}}(1+\delta_t v_t+\ldots),\\ T &= (T_t+\ldots)+(M\delta)^2 W_t+\ldots, \quad (M\delta)^2 \ll 1,\\ p &= 1+(M\delta)^2 p_t+\ldots, \end{aligned} \tag{A 2}$$

where δ_t is given by $$\delta_t = 1/R_L^{\frac{1}{2}}\delta = 1/M^{1+\omega}\tau_b^{\frac{1}{2}} \ll 1. \tag{A 3}$$

The leading terms in the equations of motion for the VTL, based upon the above representations, are

$$\frac{\partial}{\partial\eta_t}\left(\frac{v_t-\eta_t}{T_t}\right)+\frac{2}{T_t}+2\xi_t\frac{\partial}{\partial\xi_t}\left(\frac{1}{T_t}\right) = 0, \quad \frac{\partial p_t}{\partial\eta_t} = 0 \Rightarrow p_t = P_t(\xi_t) = (3\gamma A_k^2/8)\,\xi_t^{-1},$$

$$(A_k^2/2)\left[\left(\frac{v_t-\eta_t}{T_t}\right)\frac{\partial u_t}{\partial\eta_t}+\frac{2\xi_t}{T_t}\frac{\partial u_t}{\partial\xi_t}\right] = \frac{\partial}{\partial\eta_t}\left(T_t^\omega\frac{\partial u_t}{\partial\eta_t}\right),$$

$$(\sigma A_k^2/2)\left[\left(\frac{v_t-\eta_t}{T_t}\right)\frac{\partial T_t}{\partial\eta_t}+\frac{2\xi_t}{T_t}\frac{\partial T_t}{\partial\xi_t}\right] = \frac{\partial}{\partial\eta_t}\left(T_t^\omega\frac{\partial T_t}{\partial\eta_t}\right). \tag{A 4}$$

If u_t, v_t, and T_t have the forms

$$u_t = U_t(\eta_t), \quad v_t = V_t(\eta_t), \quad T_t = \Theta_t(\eta_t), \tag{A 5}$$

then the continuity, longitudinal momentum, and energy equations of (A 4) may be reduced to the following system of ordinary differential equations:

$$\frac{d}{d\eta_t}\left(\frac{V_t-\eta_t}{\Theta_t}\right)+\frac{2}{\Theta_t} = 0,$$

$$\frac{d}{d\eta_t}\left(\Theta_t^\omega\frac{dU_t}{d\eta_t}\right) = (A_k^2/2)\left(\frac{V_t-\eta_t}{\Theta_t}\right)\frac{dU_t}{d\eta_t},$$

$$\frac{d}{d\eta_t}\left(\Theta_t^\omega\frac{d\Theta_t}{d\eta_t}\right) = (\sigma A_k^2/2)\left(\frac{V_t-\eta_t}{\Theta_t}\right)\frac{d\Theta_t}{d\eta_t}. \tag{A 6}$$

The asymptotic solutions of (A 6) at the inner edge of the VTL, where $\eta_t \to -\infty$, for $(1-\omega) > 0$, are

$$\begin{aligned} \Theta_t &= \Theta_{t,-\infty}(-\eta_t)^{2/(1+\omega)}+\ldots \Rightarrow T = \Theta_{t,-\infty}(-\eta_t)^{2/(1+\omega)}+\ldots,\\ U_t &= U_{t,-\infty}(-\eta_t)^q+\ldots \quad\Rightarrow u = 1+\theta_k\delta_t^q U_{t,-\infty}(-\eta_t)^q+\ldots,\\ V_t &= V_{t,-\infty}(-\eta_t)+\ldots \quad\Rightarrow v = \delta(A_k/2)\,\xi_t^{-\frac{1}{2}})\,[1-\delta_t V_{t,-\infty}(-\eta_t)+\ldots], \end{aligned} \tag{A 7a}$$

where $$\Theta_{t,-\infty} = \left[2\left(\frac{1+\omega}{1-\omega}\right)\left(\frac{\sigma A_k^2}{2}\right)\right]^{1/(1+\omega)},$$

$$U_{t,-\infty} = \text{undetermined const.},$$

$$V_{t,-\infty} = \left[2\left(\frac{1+\omega}{1-\omega}\right)+1\right]. \tag{A 7b}$$

From a comparison of (A 7) and (6.10), it is evident that the VTL solutions match to the PVL solutions (with $\Gamma = 1$).

The asymptotic solutions of (A 6) at the outer edge of the VTL, where $\eta_t \to \infty$, are

$$\begin{aligned} \Theta_t &= 1+\Theta_{t,\infty}\,\mathrm{erfc}\,\{(\sigma/2)^{\frac{1}{2}}A_k\eta_t\}+\ldots \Rightarrow T = 1+\Theta_{t,\infty}\mathrm{erfc}\,\{(\sigma/2)^{\frac{1}{2}}A_k\eta_t\}+\ldots,\\ U_t &= U_{t,\infty}\,\mathrm{erfc}\,\{(\tfrac{1}{2})^{\frac{1}{2}}A_k\eta_t\}+\ldots \qquad \Rightarrow u = 1+\theta_k\delta_t^q\,\mathrm{erfc}\,\{(\tfrac{1}{2})^{\frac{1}{2}}A_k\eta_t\}+\ldots,\\ V_t &= -\eta_t+\ldots \qquad\qquad \Rightarrow v = \delta(A_k/2)\,\xi_t^{-\frac{1}{2}}[1-\delta_t\eta_t+\ldots], \end{aligned} \qquad \text{(A 8)}$$

where $\Theta_{t,\infty}$, $U_{t,\infty}$ = undetermined consts. A comparison of (A 8) and (5.12) confirms that the VTL solutions match to the IIL solutions.

It should be noted that, in the preceding formulation of the VTL, a requirement is that $\delta^2 \ll \theta_k\delta_t^q$. This inequality may be restated as

$$K_a = (M\delta)^2 \ll M^2\theta_k\delta_t^q = (1/M^{1+\omega}\tau_b^{\frac{1}{2}})^{(1-\sigma)/\sigma}, \qquad \text{(A 9)}$$

with $(1/M^{1+\omega}\tau_b^{\frac{1}{2}}) \ll 1$ (cf. (7.17*a*)). For $\sigma = 1$, (A 9) reduces to $K_a \ll 1$; however, for $\sigma < 1$, (A 9) reduces to $K_a \ll (1/M^{1+\omega}\tau_b^{\frac{1}{2}})^{(1-\sigma)/\sigma} \ll 1$, a more severe restriction on the theory. In turn, the restriction of (A 9) requires that δ_b must now satisfy

$$\delta_b \ll (M^{1+\omega}/R_L^{\frac{1}{2}})\exp\{-(M^{2(2+\omega)}/R_L)(M^2/R_L)^{-(1-\sigma)/(1+\sigma)}\}. \qquad \text{(A 10)}$$

For $\sigma = 1$, (A 10) reduces to (7.18); for $\sigma < 1$, (A 10) puts a more severe restriction on δ_b than does (7.18).

REFERENCES

Bush, W. B. 1968 *The RAND Corp Memo*, no. RM-5699-ARPA.

Bush, W. B. & Cross, A. K. 1967 *J. Fluid Mech.* **29**, 349.

Cole, J. D. 1968 *Perturbation Methods in Applied Mathematics*. Waltham, Mass.: Blaisdell.

Probstein, R. F. 1955 *Brown Univ. Rep.* no. TR AF 2798/1.

Probstein, R. F. & Elliot, D. 1956 *J. Aero. Sci.* **23**, 208.

Stewartson, K. 1964 *Phys. Fluids*, **7**, 667.